Torsion Units in Integral Group Rings

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Jahrestagung DFG Schwerpunkt 1489
Osnabrück
September 28th - October 2nd 2015

Notation and Setting

- G finite group
- $\mathbb{Z}G$ integral group ring over G
- Augmentation map: $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$, $\varepsilon(\sum_{g\in G} z_g g) = \sum_{g\in G} z_g$
- $V(\mathbb{Z}G)$ group of units of augmentation 1, aka normalized units
- All units of $\mathbb{Z}G$ are of the form $\pm V(\mathbb{Z}G)$, so it suffices to condider $V(\mathbb{Z}G)$

Questions

How can one construct generic units in $V(\mathbb{Z}G)$ and generate big subgroups in $V(\mathbb{Z}G)$?
$\rightarrow$ E. Jespers, Á. del Río: Group Ring Groups, De Gruyter 2015.

General Question: How is the torsion part of $V(\mathbb{Z}G)$ connected to the group base G?

E.g.: Is every finite subgroup of $V(\mathbb{Z}G)$ isomorphic to a subgroup of G? (No, Hertweck '97)
Do the orders of torsion elements of $V(\mathbb{Z}G)$ and G coincide? (Unknown. Yes, if G is solvable (Hertweck '07).)

In general we know: $X \leq V(\mathbb{Z}G)$ finite, then $|X|$ divides $|G|$ (Zhmud-Kurennoi '67).
Moreover $exp(V(\mathbb{Z}G)) = exp(G)$ (Cohn-Livingstone '65).

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Zassenhaus Conjecture and Prime Graph Question

Main open question concerning torsion units:

(First) Zassenhaus Conjecture (H.J. Zassenhaus, in the '60s)

(ZC) For $u \in V(\mathbb{Z}G)$ of finite order there exist an unit $x \in \mathbb{Q}G$ and $g \in G$ s.t. $x^{-1}ux = g$.

If such x and g exist, one says that u is rationally conjugate to g.

Prime Graph Question (Kimmerle, 2006)

(PQ) Let p and q be different primes. If $V(\mathbb{Z}G)$ contains an element of order pq, does G contain an element of order pq?

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Some known results on ZC

The Zassenhaus Conjecture is known to hold for

- Abelian Groups (Higman '39)
- Nilpotent Groups (Weiss '91)
- Groups with normal Sylow subgroup with abelian complement (Hertweck '07)
- Metacyclic Groups (Hertweck '08)
- Cyclic-By-Abelian Groups (Caicedo-M'-del Rio '13)
- Two other special series of metabelian groups (Sehgal-Weiss '86, Marciniak-Ritter-Sehgal-Weiss '87)
- Groups till order 71 (Höfert '04)
- $\mathrm{PSL}(2, q)$ for $q \leq 25$ (Luthar-Passi, Hertweck, Gildea, Kimmerle-Konovalov, Bächle-M')
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Theorem (Kimmerle-Konovalov, 2012)

Suppose that **(PQ)** has an affirmative answer for each almost simple image of G, then it has also a positive answer for G.

(G almost simple, if $S \leq G \leq \mathrm{Aut}(S)$ for some non-ab. simple S.)

Keeping this in mind (PQ) is known for:

- Solvable groups (Kimmerle '06)
- Groups whose order is divisible by at most three different primes (Kimmerle-Konovalov '12, Bächle-M' '14)
- Almost simple groups whose socle is
  - In the first half of the sporadic simple groups (Bovdi-Konovalov et al. '07 -'12)
  - $\mathrm{PSL}(2, p)$, with p a prime (Hertweck '08, M' '14)
  - An alternating group of degree up to 16 (Luthar-Passi, Hertweck, Salim, Bächle-Caicedo '15)

Most of these results rely on the so-called HeLP-Method.

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HeLP-Package

The HeLP-method, named after Hertweck, Luthar and Passi (name due to A. Konovalov) uses ordinary and modular characters to tackle these questions. It has now been implemented into a GAP-package, available at

http://homepages.vub.ac.be/abachle/help/

Two main purposes:
Reserachers working in the field can use it to obtain new results.
Results using the method can be checked by readers.

HeLP-Method: Partial Augmentations

Let x^G be a conjugacy class in G and $u = \sum_{g \in G} z_g g \in \mathbb{Z}G$. Then

$$\varepsilon_x(u) = \sum_{g \in x^G} z_g$$

is called the **partial augmentation** of u with respect to x.

Theorem (Marciniak-Ritter-Sehgal-Weiss '87)

$u \in V(\mathbb{Z}G)$ is rationally conjugate to an element of G if and only if $\varepsilon_x(u^d) \geq 0$ for all $x \in G$ and divisors d of n.

Theorem (Higman '39, Berman '53)

If $u \in V(\mathbb{Z}G)$ is a torsion unit, then $\varepsilon_1(u) = 0$ or $u = 1$.

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Example: $u \in V(\mathbb{Z}A_5)$, $o(u) = 2 \cdot 5$ (Luthar-Passi, 1989)

	1a	2a	3a	5a	5b
χ	4	0	1	-1	-1

, $\mathbb{Z}(\chi) = \mathbb{Z}$.

- $o(u^5) = 2$, $\chi(u^5) = \varepsilon_{2a}(u^5)\chi(2a) = 0$

$$D(u^5) \sim \mathrm{diag}(1, 1, -1, -1)$$

- $o(u^6) = 5$,
$\chi(u^6) = \varepsilon_{5a}(u^6)\chi(5a) + \varepsilon_{5b}(u^6)\chi(5b) = (\varepsilon_{5a}(u^6) + \varepsilon_{5b}(u^6)) \cdot (-1) = -1$

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The HeLP-Inequalities

Theorem (Luthar-Passi '89; Hertweck '04)

Let

- $u \in \mathbb{Z}G$ *torsion unit of order* n,
- F *splitting field for* G *with* $p = \mathrm{char}(F) \nmid n$,
- χ *a (p-Brauer) character of an* F*-representation* D *of* G,
- $\zeta \in \mathbb{C}$ *primitive n-th root of unity,*
- $\xi \in F$ *corresponding n-th root of unity.*

Multiplicity $\mu_\ell(u, \chi, p)$ *of* ξ^ℓ *as an eigenvalue of* $D(u)$ *is given by*

$$\frac{1}{n} \sum_{d|n} \mathrm{Tr}_{\mathbb{Q}(\zeta^d)/\mathbb{Q}}(\chi(u^d)\zeta^{-d\ell}) \in \mathbb{Z}_{\geq 0}.$$

This yields a system of inequalities for the partial augmentations $\varepsilon_x(u)$ of u, assuming knowledge on the partial augmentations of the powers u^d for divisors d of the order of u:

$$\frac{1}{n}\sum_{d|n} \mathrm{Tr}_{\mathbb{Q}(\zeta^d)/\mathbb{Q}}(\chi(u^d)\zeta^{-d\ell})$$
$$=\frac{1}{n}\sum_{\substack{d|n\\ d\neq 1}} \mathrm{Tr}_{\mathbb{Q}(\zeta^d)/\mathbb{Q}}(\chi(u^d)\xi^{-d\ell}) + \frac{1}{n}\sum_{\substack{x^G:\ x \text{ is}\\ p-\text{regular}}} \varepsilon_x(u)\mathrm{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\chi(x)\xi^{-\ell})$$

In a less technical way: For every (p-Brauer) character χ of G and character ψ of $\langle u \rangle$ we have

$$\langle \chi|_{\langle u \rangle}, \psi \rangle_{\langle u \rangle} \in \mathbb{Z}_{\geq 0}.$$

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$$\langle \chi|_{\langle u\rangle}, \psi\rangle_{\langle u\rangle} \in \mathbb{Z}_{\geq 0}.$$

In the previous example, $u \in V(\mathbb{Z}A_5)$, $o(u) = 2 \cdot 5$, this yields:

$$\mu_0(u, \chi, 0) = -2/5\,(\varepsilon_{5a}(u) + \varepsilon_{5b}(u))$$
$$\mu_1(u, \chi, 0) = -1/10\,(\varepsilon_{5a}(u) + \varepsilon_{5b}(u)) + 1/2$$
$$\mu_2(u, \chi, 0) = 1/10\,(\varepsilon_{5a}(u) + \varepsilon_{5b}(u)) + 1/2$$
$$\mu_5(u, \chi, 0) = 2/5\,(\varepsilon_{5a}(u) + \varepsilon_{5b}(u))$$

As it is easily seen, the above expressions can not be all non-negative integers.

Generic Results

For series of groups possesing generic character tables computing motivating examples can lead to generic results. E.g.:

- For $G = \mathrm{PSL}(2, p^f)$ and $r \neq p$ a prime, elements of r-power order in $V(\mathbb{Z}G)$ are rationally conjugate to elements of G. (M' '14)
- For $G = \mathrm{PSL}(2, p)$ and $r \neq p$ an odd prime, there is exactly one non-trivial possibility for partial augmentations of elements of order $2r$ satisfying the HeLP-constraints. (del Río-Serrano '15)
- Some generic results for alternating and symmetric groups were recently obtained by Bächle-Caicedo.

Application: Groups of small order (Krauß-Meyer-Ritter)

Group Order	GAP-Id and result of HeLP-method investigating ZC
72	15, 22, 23, 24, 31, 33, 35, **40(6)**, 42, 43, 44
96	13, 30, 31, 40, 41, 43, 64, **65(8)**, 66, **67(8)**, 76, 77, 81-83, 85, 86, 89-93, 95-97, 101-105, 131, 136, 137, 142-147, 151, 153, 154, 160, **185(4)**, **186(4)**, **187(4)**, 188-193, **194(4)**, **195(4)**, 226 , **227(2,4)**
120	5, 35, 37, 38, 39
144	31, 32, **33(4)**, 54-62, 64-67, 86-89, 91, 93-100, 109, **115(6,12)**, 116, **117(6)**, **118(12)**, **119(12)**, 120, 121, 122, **123(4,12)**, 124, 125, **126(4)**, 127-129, 131, 133, 135-142, 144, 145, 147, 148, 150-154, 168, 170-175, 177, **182(6)**, 183, **186(6)**, 187, 188-190
160	13, 30, 31, 40, 41, 43, 74, 80, 86, 88, 90, 91, 95-97, 99, 100, 103-105, 106, 107, 109-111, 115-119, 145, 150, 151, 156, 157, 158-160, 161, 165, 167, 168, 174, **234(2,4)**
168	23, **43(6)**, 45, 46, 48, 49, 53
180	17, 19, 22, 24, 25, 27, 29, 30, 36
192*	**180(4,8)**, **181(4,8), 182(4)**, **183(8)**, 184, **185(4)**, 944-954, **955(2,4)**, 956, **957(4)**, **958(4,8)**, **959(4,8)**, **960(4,8)**, **961(4)**, **962(8)**, **963(8)**, **964(8)**, **965(8)**, **966(8)**, **967(8)**, **968(8)**, **969(4)**, **970(4)**, **971(4)**, **972(4)**, **973(4,8)**, **974(4,8)**, **975(4,8)**, **976(4,8)**, 977-980, **981(8)**, **982(8)**, 983-986, **987(4,8)**, **988(8)**,**989(8)**, **990(4,8)**, **991(4)**, **1468(4)**, **1469(4)**, **1470(4)**, **1471(4)**, **1472(4)**, **1473(4)**, 1474-1476, **1477(4)**, **1478(4)**, 1479-1486 , **1487(4)**, **1488(4)**, **1495(2,4)**, **1538(2,4)** (* not all necessary groups tested)
200	24, 25, 26, 32, 34, 36, **43(10)**

Application: PQ for 4-primary groups

The simple groups with four different prime divisors:

$\mathrm{PSL}(2, p)$	A_7	$\mathrm{PSL}(3, 4)$	$\mathrm{PSU}(3, 8)$	$\mathrm{P}\,\Omega_+(8, 2)$
$\mathrm{PSL}(2, 2^f)$	A_8	$\mathrm{PSL}(3, 5)$	$\mathrm{PSU}(3, 9)$	$\mathrm{Sz}(8)$
$\mathrm{PSL}(2, 3^f)$	A_9	$\mathrm{PSL}(3, 7)$	$\mathrm{PSU}(4, 3)$	$\mathrm{Sz}(32)$
	A_{10}	$\mathrm{PSL}(3, 8)$	$\mathrm{PSU}(5, 2)$	$G_2(3)$
	$\mathrm{PSL}(2, 16)$	$\mathrm{PSL}(3, 17)$	$\mathrm{PSp}(4, 4)$	${}^3D_4(2)$
	$\mathrm{PSL}(2, 25)$	$\mathrm{PSL}(4, 3)$	$\mathrm{PSp}(4, 5)$	${}^2F_4(2)'$
	$\mathrm{PSL}(2, 27)$	$\mathrm{PSU}(3, 4)$	$\mathrm{PSp}(4, 7)$	M_{11}
	$\mathrm{PSL}(2, 49)$	$\mathrm{PSU}(3, 5)$	$\mathrm{PSp}(4, 9)$	M_{12}
	$\mathrm{PSL}(2, 81)$	$\mathrm{PSU}(3, 7)$	$\mathrm{PSp}(6, 2)$	J_2

HeLP-method   Lattice-method   No method yet

How much is possible?

There are some natural bounds for the package:

- Not all known characters are available in the GAP-Library yet. For this reason our package allows using any class function, entered e.g. manually or obtained by inducing of characters from subgroups.
- The main computations always concern solving the inequalities. So far we use for that purpose:
  - The `zsolve`-function from `4ti2` (Walter) and the `4ti2-Interface` for GAP (Gutsche) to solve the inequalities.
  - The `redund`-function from `lrs` (Avis) to reduce the systems to smaller sizes. (Its behaviour with `zsolve` is sometimes strange.)

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