Nilpotent associative algebras and coclass theory

Bettina Eick and Tobias Moede

TU Braunschweig

SPP Jahrestagung, Osnabrück, Sep. 2015

Introduction

Associative Algebras

Associative Algebras

An associative algebra over a field $\mathbb{F}$ is a vectorspace over $\mathbb{F}$ equipped with an associative multiplication.

Identity

It is NOT assumed that an associative algebra contains an identity element.

Examples

Examples

(1) Matrix algebras: subalgebras of $M_n(\mathbb{F})$, for example the upper triangular matrices with 0's on the diagonal.

(2) Group algebras: $\mathbb{F}G$ where G is a finite group; these algebras always have an identity.

(3) Quaternion algebras: certain 4-dimensional algebras; these also have an identity.

Nilpotency

Nilpotency

An associative algebra A is *nilpotent* if there exists $c \in \mathbb{N}$ so that every product of $c+1$ elements in A is zero. The smallest c with this property is the *class* $cl(A)$ of A.

Power Ideals

For an algebra A let A^i be the ideal spanned by all products of i elements in A. Then A is nilpotent of class c if and only if

$$A = A^1 > A^2 > \ldots > A^c > A^{c+1} = \{0\}.$$

Coclass

Coclass

Let A be a finite dimensional nilpotent associative algebra. Then the *coclass* of A is defined as

$$dim(A) - cl(A).$$

Examples I

Example

Let A be the subalgebra of $M_n(\mathbb{F})$ consisting of all upper triangular matrices with 0's on the diagonal.

(1) A is nilpotent of dimension $n(n-1)/2$ and class $n-1$.

(2) A has coclass $n(n-1)/2 - (n-1) = (n-1)(n-2)/2$.

Examples II

Example

Let G be a finite p-group of order p^n, $\mathbb{F}$ a finite field of characteristic p and $A = J(\mathbb{F}G)$.

(1) A is nilpotent of dimension $p^n - 1$ and class $p^l - 1$, where l is the length of the Jennings series of G.

(2) A has coclass $p^n - 1 - (p^l - 1) = p^n - p^l$.

Significance

Structure theory (Wedderburn/Jacobson)

Let A be a finite dimensional associative algebra with identity.

(1) $A/J(A)$ is a direct sum of full matrix algebras over skewfields.

(2) $J(A)$ is a nilpotent associative algebra.

Classification

General Aims

A wide open problem

Classify the finite dimensional nilpotent associative algebras over a field $\mathbb{F}$ up to isomorphism.

Using the dimension

Dimension

Classify the nilpotent associative algebras over $\mathbb{F}$ of dimension d:

- $d = 1$ is trivial:
  there is just one such algebra $C_1 = \langle a \mid a^2 = 0 \rangle$.
- $d = 2$ is easy:
  there are two such algebras $C_1 \oplus C_1$ and $C_2 = \langle a \mid a^3 = 0 \rangle$.
- $d = 3$ is known (Willem de Graaf), but not easy:
  If $\mathbb{F}$ is infinite, then there are infinitely many algebras.
  If $\mathbb{F}$ is finite, then there are $|\mathbb{F}| + 6$ or $|\mathbb{F}| + 5$ algebras.

Proceed with this? — Seems daunting.

Open problem

Higman

The number of isomorphism types of algebras of dimension n over $\mathbb{F}_q$ is PORC (Polynomial on residue classes).

Open Problem

Is the number of isomorphism types of nilpotent associative algebras of dimension n over $\mathbb{F}_q$ a PORC function?

Using the coclass

Question

Is it possible to classify the finite dimensional nilpotent associative algebras over $\mathbb{F}$ of coclass r?

Example: Coclass $r = 0$

This is easy! The resulting algebras are $C_n := \langle a \mid a^{n+1} = 0 \rangle$ for $n \in \mathbb{N}_0$.

Sounds promising?

Why coclass?

Nilpotent Groups

Coclass theory has first been considered for finite p-groups, initiated by Leedham-Green and Newman. Result is a very useful structure theory!

Nilpotent Lie Algebras

It has also been considered for nilpotent Lie algebras, mainly due to Shalev and Zelmanov.

Nilpotent Associative Algebras

It seems a promising approach for associative algebras. The main aim of this DFG project was to investigate this.

Coclass Theory

Coclass Graph

The coclass graph

Let $\mathbb{F}$ be a field and $r \in \mathbb{N}_0$. The graph $\mathcal{G}_{\mathbb{F}}(r)$ is defined by:

- Vertices correspond one-to-one to isomorphism types of finite dimensional nilpotent associative $\mathbb{F}$-algebras of coclass r;
- There is an edge $A \to B$ if $A \cong B/B^{cl(B)}$; that is, if B is a *descendant* of A.

Examples

Small Coclass

- $\mathcal{G}_{\mathbb{F}}(0)$ is easy for all fields $\mathbb{F}$.
- $\mathcal{G}_{\mathbb{F}}(1)$ is a tree with root $C_1 \oplus C_1$.
- $\mathcal{G}_{\mathbb{F}}(2)$ is again more complicated...

Observations

Observation

In these small examples there are always finitely many infinite paths starting from the root.

First step

Investigate the infinite paths in $\mathcal{G}_{\mathbb{F}}(r)$!

Infinite paths

Pro-nilpotent algebras

Let $A_1 \to A_2 \to \ldots$ be an infinite path in $\mathcal{G}_{\mathbb{F}}(r)$ and let A be the inverse limit of this path. Then

(a) A is an infinite dimensional associative algebra.

(b) $A/A^{i+cl(A)} \cong A_i$ for all large enough i; say $cc(A) = r$.

(c) Equivalent paths define isomorphic inverse limits.

Correspondence

The maximal infinite paths in $\mathcal{G}_{\mathbb{F}}(r)$ correspond one-to-one to the isomorphism types of infinite dimensional pro-nilpotent associative $\mathbb{F}$-algebras of coclass r.

Some definitions

Formal power series

(a) Let $\mathbb{F}[[t]]$ be the ring of formal power series over $\mathbb{F}$.

(b) Let $\mathbb{F}_o[[t]]$ be the ideal generated by t in $\mathbb{F}[[t]]$.

Annihilators

For an algebra A is

(a) $Ann(A) = \{a \in A \mid ab = ba = 0 \text{ for all } b \in A\}$.

(b) $Ann_0(A) = \{0\}$ and $Ann_i(A) = Ann(A/Ann_{i-1}(A))$ for $i \geq 1$.

(c) $Ann_*(A) = \cup_{i \in \mathbb{N}} Ann_i(A)$.

A structure theorem

The following theorem exhibits the structure of the inverse limits of the infinite paths in $\mathcal{G}_{\mathbb{F}}(r)$.

Theorem (Eick & Moede)

Let $\mathbb{F}$ be a field and $r \in \mathbb{N}_0$.
A is isomorphic to the inverse limit of an infinite path in $\mathcal{G}_{\mathbb{F}}(r)$ if and only if $dim(Ann_*(A)) = r$ and $A = Ann_*(A) \rtimes \mathbb{F}_0[[t]]$.

In other words

Let $\mathbb{F}$ be a field and $r \in \mathbb{N}_0$.
Each infinite path in $\mathcal{G}_{\mathbb{F}}(r)$ can be constructed as split extension of an r-dimensional nilpotent algebra with $\mathbb{F}_o[[t]]$.

Application

Numbers

Let $n_{\mathbb{F}}(r)$ denote the number of maximal infinite paths in $\mathcal{G}_{\mathbb{F}}(r)$.

(a) $n_{\mathbb{F}}(0) = 1$ for all fields $\mathbb{F}$.

(b) $n_{\mathbb{F}}(1) = 1$ for all fields $\mathbb{F}$.

(c) $n_{\mathbb{F}}(2) = \infty$ if $\mathbb{F}$ is infinite and $n_{\mathbb{F}}(2) = |\mathbb{F}| + 4$ if $\mathbb{F}$ is finite.

Theorem (Eick & Moede)

$n_{\mathbb{F}}(r)$ is finite if and only if $r \leq 1$ or $\mathbb{F}$ is finite.

Algorithms

Descendants

Descendants

An associative algebra B is a descendant of A if $A \cong B/B^{cl(A)+1}$.

Descendant tree

Given A in $\mathcal{G}_{\mathbb{F}}(r)$ we denote with $\mathcal{T}_A$ the full subtree of $\mathcal{G}_{\mathbb{F}}(r)$ of descendants of A.

Maximal descendant tree

A descendant tree $\mathcal{T}_A$ is maximal if it is not properly contained in another descendant tree; that is, if $A = \{0\}$ or $A/A^{cl(A)}$ has coclass smaller than r.

Algorithm I

Immediate descendants

Let $\mathbb{F}$ be a finite field. Developed an effective algorithm to determine up to isomorphism all immediate descendants of an algebra A (in $\mathcal{G}_{\mathbb{F}}(r)$).

Ingredients

(a) Compute $Aut(A)$.

(b) Compute the multiplication M and the nucleus N of A.
(N is a subspace of the finite dimensional vectorspace M.)

(c) Compute the natural action of $Aut(A)$ on M.

rm (d) Compute orbits and stabilizers of all supplements to N in M.

Algorithm II

Theorem (Eick & Moede)

Let $\mathbb{F}$ be a finite field. Then $\mathcal{G}_{\mathbb{F}}(r)$ consists of finitely many maximal descendant trees. The roots of these trees have dimension at most $2r$.

Roots

Let $\mathbb{F}$ be a finite field. Developed an effective algorithm to determine up to isomorphism the roots of $\mathcal{G}_{\mathbb{F}}(r)$.

Applications

Application

Algorithm I and II allow to investigate $\mathcal{G}_{\mathbb{F}}(r)$.

(a) Compute the roots of the maximal descendant trees.

(b) Compute iteratedly immediate descendants of these roots and their descendants.

(c) Yields large finite parts of the infinite graph $\mathcal{G}_{\mathbb{F}}(r)$.

Experiments

Experiments

Determined finite parts of $\mathcal{G}_{\mathbb{F}}(r)$ for many finite fields and various r.

Result

Many detailed insights into the structure of $\mathcal{G}_{\mathbb{F}}(r)$.

Periodic patterns

Coclass trees

Coclass trees

A descendant tree $\mathcal{T}_A$ in $\mathcal{G}_{\mathbb{F}}(r)$ is a *coclass tree* if it has a unique infinite path.

Maximal coclass trees

A coclass tree is maximal if it is not properly contained in another coclass tree.

Theorem (Eick & Moede)

Let $\mathbb{F}$ be a finite field and $r \in \mathbb{N}_0$. Then $\mathcal{G}_{\mathbb{F}}(r)$ consists of finitely many maximal coclass trees and finitely many other vertices.

Periodicity

Periodicity

Let $\mathcal{T}$ be a maximal coclass tree with root A and infinite path $A = A_1 \to A_2 \to \ldots$.

(a) $\mathcal{T}$ is *virtually periodic* with period d and periodic root A_l if $\mathcal{T}_{A_i}$ and $\mathcal{T}_{A_{i+d}}$ are graph isomorphic for each $i \geq l$.

(b) $\mathcal{T}$ has *depth* k if every vertex in $\mathcal{T}$ has distance at most k from the infinite path.

Conjectures

Conjecture (Eick & Moede)

Let $\mathbb{F}$ be a finite field and $r \in \mathbb{N}_0$. Then each maximal coclass tree $\mathcal{T}$ in $\mathcal{G}_{\mathbb{F}}(r)$ is virtually periodic and has finite depth.

Conjecture (Eick & Moede)

Let $\mathbb{F}$ be a finite field and $r \in \mathbb{N}_0$. The infinitely many algebras in $\mathcal{G}_{\mathbb{F}}(r)$ can be described by finitely many parametrised presentations.

Implications

If the conjectures hold, then the infinitely many nilpotent associative $\mathbb{F}$-algebras of coclass r can be classified!

Coclass 1

Conjecture (Eick & Moede)

Let $\mathbb{F}$ be a finite field. Then $\mathcal{G}_{\mathbb{F}}(1)$ consists of a single coclass tree. This is periodic with period $|\mathbb{F}| - 1$ and depth 1.

Coclass 2

Conjecture (Eick & Moede)

Let $\mathbb{F}$ be a finite field of char $p > 2$ and size q. Then $\mathcal{G}_{\mathbb{F}}(2)$ consists of $3q + 6$ maximal descendant trees. Of these, $2q + 2$ are finite and $q + 4$ are maximal coclass trees. The maximal coclass trees are all virtually periodic with

(a) There are $q + 1$ maximal coclass trees of depth 1 and period $q - 1$;

(b) There is 1 maximal coclass tree of depth 1 and period 1;

(c) There are 2 maximal coclass trees of depth 2 and period $p(q - 1)$;

Coclass 2

Conjecture (Eick & Moede)

Let $\mathbb{F}$ be a finite field of char $p = 2$ and size q. Then $\mathcal{G}_{\mathbb{F}}(2)$ consists of $3q + 5$ maximal descendant trees. Of these, $2q + 1$ trees are finite and $q + 4$ are maximal coclass trees. The maximal coclass trees are all virtually periodic with

(a) There are $q + 3$ maximal coclass trees of depth 1 and period $q - 1$;

(b) There is 1 maximal coclass tree of depth 2 and period $p(q - 1)$;