Gröbner Bases over Algebraic Number Fields

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joint work with
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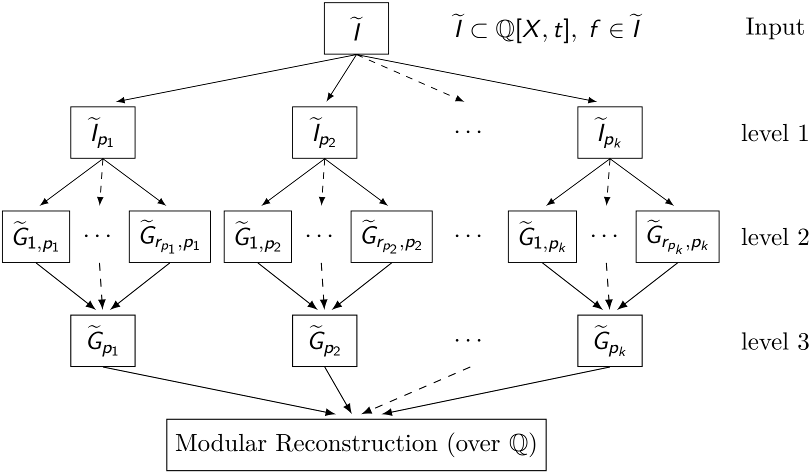
Problem

Given an ideal $I \subseteq K[x_1, \ldots, x_n]$ where $K = \mathbb{Q}(\alpha)$ is a number field, what is an efficient way to compute a Gröbner basis of I?

Overview of the New Method



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- fix a global product ordering $\succ_K := (\succ_1, \succ_2)$ on $\mathrm{Mon}(X, t)$; this is an elimination ordering w.r.t. X;
- for a polynomial $q \in S$ and a set $G \subseteq S$, we write:

  $\mathsf{lm}(q)$: the *leading monomial* of q,
  $\mathsf{Lm}(G)$: the *set of leading monomials* of the elements in G.

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Theorem

Let $\widetilde{G}$ be the reduced Gröbner basis of $\widetilde{I}$ w.r.t. $\succ_K$. Then, if $\widetilde{I} \neq \langle 1 \rangle$, we have $f \in \widetilde{G}$. In any case, all elements $m(X, t) \in \widetilde{G} \setminus \{f\}$ are monic if considered as elements in $\mathbb{Q}[t][X]$. Furthermore, $(\widetilde{G} \setminus \{f\})|_{t=\alpha}$ is the reduced Gröbner basis of I w.r.t. $\succ_1$.

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  - The superfluous elements yield new S-pairs which usually make the subsequent computations inefficient.
  - Noro's modification: Each generated basis element is made monic in $(\mathbb{Q}[t])[X]$ before it is added to the basis, that is, the inverse of an algebraic number is computed.
  - However, this is in general computationally expensive.

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- We use a different approach to reduce the number of basis elements which are computed before a monic element $X^a + \text{(lower terms)}$ is generated.
- Our approach makes use of
  - modular methods w.r.t. different prime numbers to avoid intermediate coefficient swell;
  - factorization of the minimal polynomial in positive characteristic to considerably reduce the degree of the field extensions.

Theorem

Let $p_1, \ldots, p_k$ be distinct prime numbers, and let $N = p_1 \cdots p_k$ be their product. Then we have the ring isomorphism:

$$\mathbb{Z}/N\mathbb{Z} \cong \mathbb{F}_{p_1} \times \ldots \times \mathbb{F}_{p_k}.$$

Two Variants of the Chinese Remainder Theorem

Theorem

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Theorem

Let $f_{1,p}, \ldots, f_{r_p,p} \in \mathbb{F}_p[t]$ *be pairwise coprime polynomials, and let* $f_p = f_{1,p} \cdots f_{r_p,p}$ *be their product. Then we have the ring isomorphism*

$$\mathbb{F}_p[t]/\langle f_p \rangle \cong \mathbb{F}_p[t]/\langle f_{1,p} \rangle \times \ldots \times \mathbb{F}_p[t]/\langle f_{r_p,p} \rangle.$$

Level 1: Compute modulo several prime numbers

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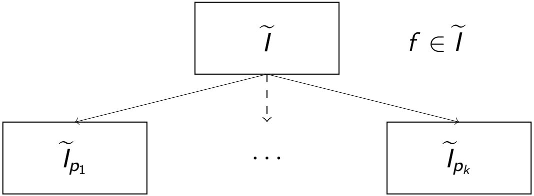

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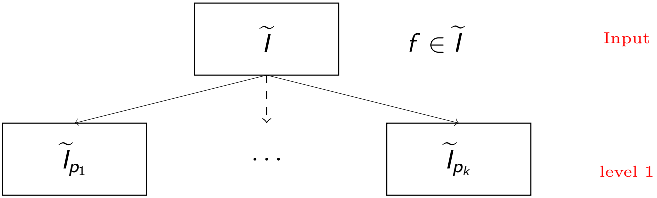

Definition

Let $f \in \mathbb{Q}[t]$ be as given above. Let p be a prime not dividing any numerator or denominator of the coefficients occurring in f. We say that p is *admissible of type* A w.r.t. f if the reduction f_p is reducible and square-free over $\mathbb{F}_p$. In this case, we write $f_p = \prod_{1 \le i \le r_p} f_{i,p}$.

Level 2: Compute modulo the factors of f_p

$$\begin{array}{ccc}
\boxed{\widetilde{I}} & f \in \widetilde{I} & \text{Input} \\
\downarrow & & \\
\boxed{\widetilde{I}_p} & & \text{level 1}
\end{array}$$

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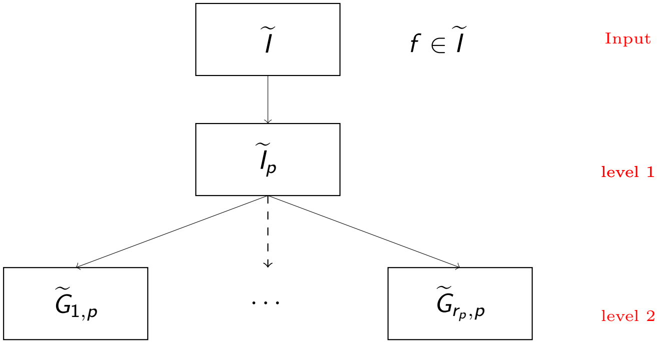

Level 2: Compute modulo the factors of f_p

$$\begin{tikzcd}
{} \arrow[d] & & \\
& \widetilde{I}_p \arrow[dl] \arrow[d, dashed] \arrow[dr] & & \text{level 1} \\
\widetilde{G}_{1,p} & \cdots & \widetilde{G}_{r_p,p} & \text{level 2}
\end{tikzcd}$$

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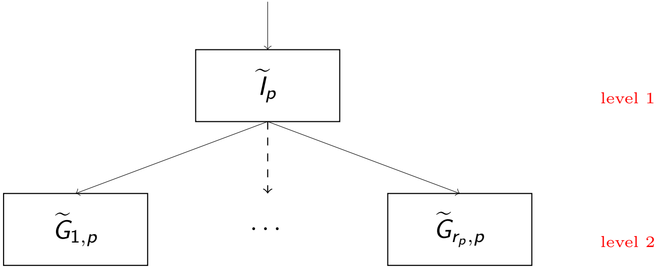

- Let $f_p = \prod_{1 \le i \le r_p} f_{i,p}$ be the irreducible factorization of f_p over $\mathbb{F}_p$, with $r_p > 1$.

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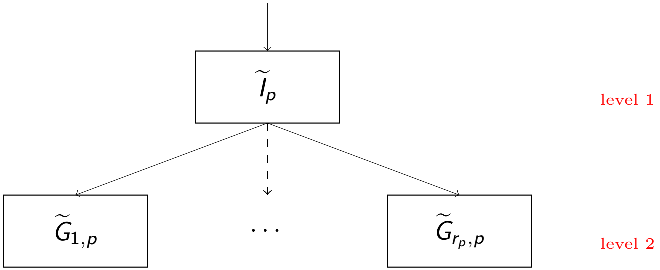

- Let $f_p = \prod_{1 \le i \le r_p} f_{i,p}$ be the irreducible factorization of f_p over $\mathbb{F}_p$, with $r_p > 1$.
- For each $i \in \{1, \dots, r_p\}$, we compute the reduced Gröbner basis $\widetilde{G}_{i,p}$ of the ideal $\widetilde{I}_{i,p} := \langle \widetilde{H}_p \cup \{f_{i,p}\} \rangle$.

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$$
\begin{tikzcd}
 & \bullet \arrow[dl] \arrow[d, dashed] \arrow[dr] & \\
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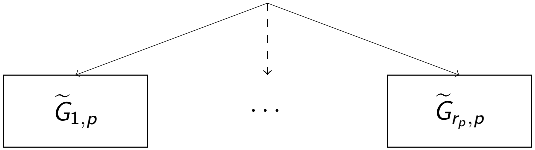

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Let p be a prime as in the previous definition. In addition, suppose that p does not divide any numerator or denominator of the coefficients occurring in $\widetilde{H}$. Then we say that p is *admissible of type* B w.r.t. f and $\widetilde{H}$ if for all indices i, j with $i \neq j$

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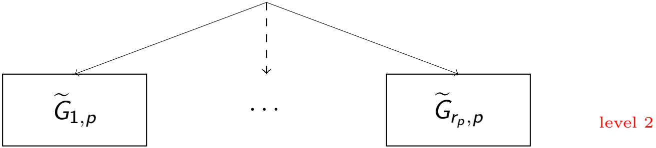

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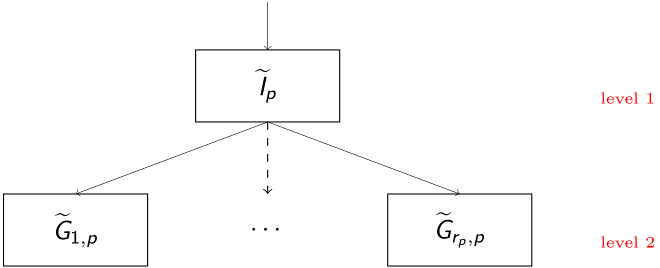
1. the sizes of $\widetilde{G}_{i,p}$ and $\widetilde{G}_{j,p}$ coincide, and
2. $\mathrm{Lm}(\widetilde{G}_{i,p} \setminus \{f_{i,p}\}) = \mathrm{Lm}(\widetilde{G}_{j,p} \setminus \{f_{j,p}\})$.

Level 3: Reconstruct $\widetilde{G}_p$



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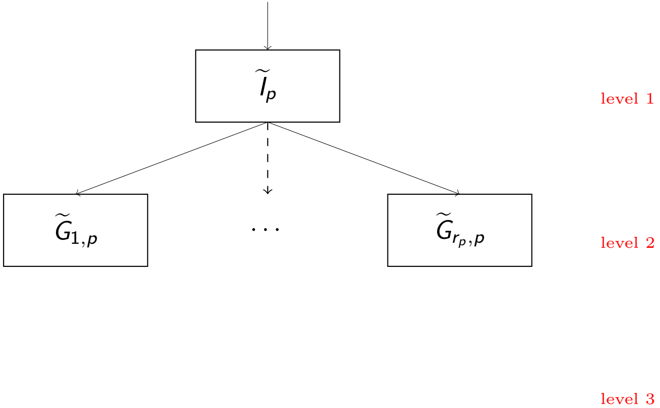

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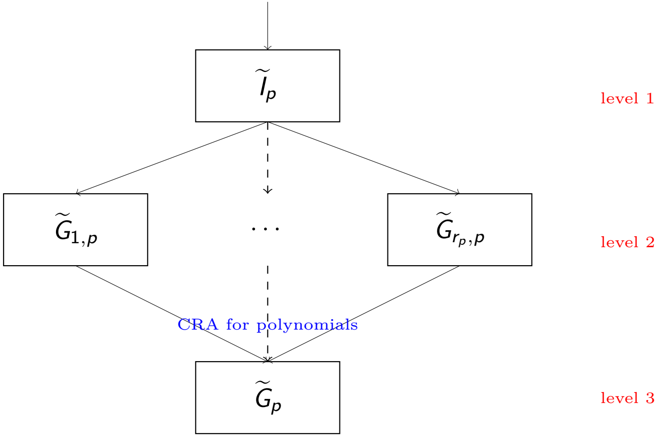

Delete Unlucky Primes

Definition

Let $\widetilde{I}$ be an ideal given as above and let p be a prime number. Furthermore, let $\widetilde{G}$ be the reduced Gröbner basis of $\widetilde{I}$ and let $\widetilde{G}_p$ be the reduced Gröbner basis of $\widetilde{I}_p$. Then p is called *lucky* for $\widetilde{I}$ if and only if $\mathsf{Lm}(\widetilde{G}_p) = \mathsf{Lm}(\widetilde{G})$. Otherwise p is called *unlucky* for $\widetilde{I}$. [Idrees, Pfister, and Steidel, 2011]

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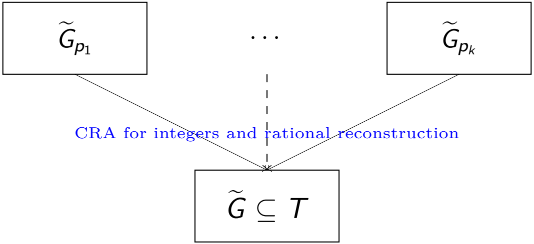
DeleteUnluckyPrimes: *If $\mathcal{P}$ is the set of selected primes, with corresponding Gröbner bases collected in $\mathcal{GP}$, define an equivalence relation on $(\mathcal{GP}, \mathcal{P})$ by*

$$(\widetilde{G}_p, p) \sim (\widetilde{G}_q, q) :\Longleftrightarrow \mathsf{Lm}(\widetilde{G}_p) = \mathsf{Lm}(\widetilde{G}_q).$$

Store the equivalence class of largest cardinality in $(\mathcal{GP}, \mathcal{P})$, and delete the others [Idrees, Pfister, and Steidel, 2011].

A Test in Positive Characteristic: pTestSB



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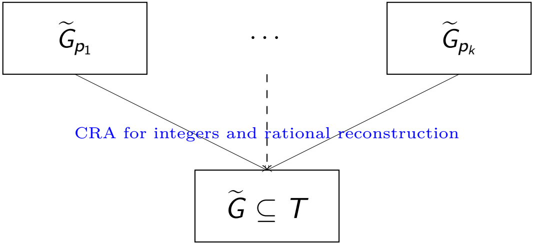

pTestSB: *We randomly choose a prime* $p \notin \mathcal{P}$ *which is admissible of type* B *w.r.t.* f *and* $\widetilde{H}$. *We test if including this prime in the set* $\mathcal{P}$ *would improve the result. That is, we explicitly test whether* $\widetilde{I}_p$ *reduces to zero w.r.t* $\widetilde{G}$ *mapped to* $\mathbb{F}_p[X, t]$, *and vice-versa, whether* $\widetilde{G}$ *mapped to* $\mathbb{F}_p[X, t]$ *reduces to zero w.r.t.* $\widetilde{G}_p$. [Idrees, Pfister, and Steidel, 2011].

Final Test

For homogeneous ideals or for local monomial orderings, we have the following result:

Theorem (Arnold 2003 and Pfister 2007)

If $\widetilde{\mathsf{I}}$ reduces to zero w.r.t. $\widetilde{\mathsf{G}}$ and if $\widetilde{\mathsf{G}}$ is the reduced Gröbner basis of $\langle \widetilde{\mathsf{G}} \rangle$, then $\widetilde{\mathsf{I}} = \langle \widetilde{\mathsf{G}} \rangle$.

Theorem

Let $\widetilde{G}$ be the reduced Gröbner basis of $\widetilde{I}$ with respect to $\succ_K$. Then $(\widetilde{G} \setminus \{f\})|_{t=\alpha}$ is the reduced Gröbner basis of I with respect to $\succ_1$.

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---

nfmodStd

Input: $I = \langle g_1(X,\alpha), \ldots, g_s(X,\alpha) \rangle \subseteq S = K[X]$.

Output: $G \subseteq S$, a Gröbner basis of I w.r.t. $\succ_1$.

1: map I to $\langle \widetilde{H} \rangle$ via the map sending α to t

2: $\widetilde{I} \longleftarrow \langle \widetilde{H} \rangle + \langle f \rangle$

3: compute the reduced Gröbner basis $\widetilde{G}$ of $\widetilde{I}$ w.r.t. $\succ_K = (\succ_1, \succ_2)$

4: lift $\widetilde{G}$ to G via the map sending t to α

5: **return** G

Implementation and Timings

- Our algorithm is implemented in SINGULAR in the library `nfmodstd.lib`.
  http://www.singular.uni-kl.de
  [Boku, Decker and Fieker, 2015].

	Magma	Singular					
deg	GB	std	modStd		nfmodStd		
			1 core	32 cores	1 core	32 cores	
2	1241.98	1.51	1.24	0.37	0.22	0.13	
5	error	70.55	19.59	4.79	1.89	0.61	
7	-	0.90	143.79	9.34	3.27	0.51	
7	-	314.00	11212.00	1118.78	97.43	19.23	
6	-	265.53	9163.38	567.03	686.01	99.41	
12	-	2061.95	3321.28	256.58	430.23	71.47	
2	2.93	8931.13	197.20	47.54	24.26	8.99	
8	-	0.90	2044.08	195.41	8.54	1.87	
7	-	15477.87	15274.97	4787.49	92.99	23.89	

GB = GroebnerBasis

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- Modulo p, we compute in rings with minimal polynomials of degree much less than $\mathsf{deg}(f)$.
- The algorithm is parallel in nature.